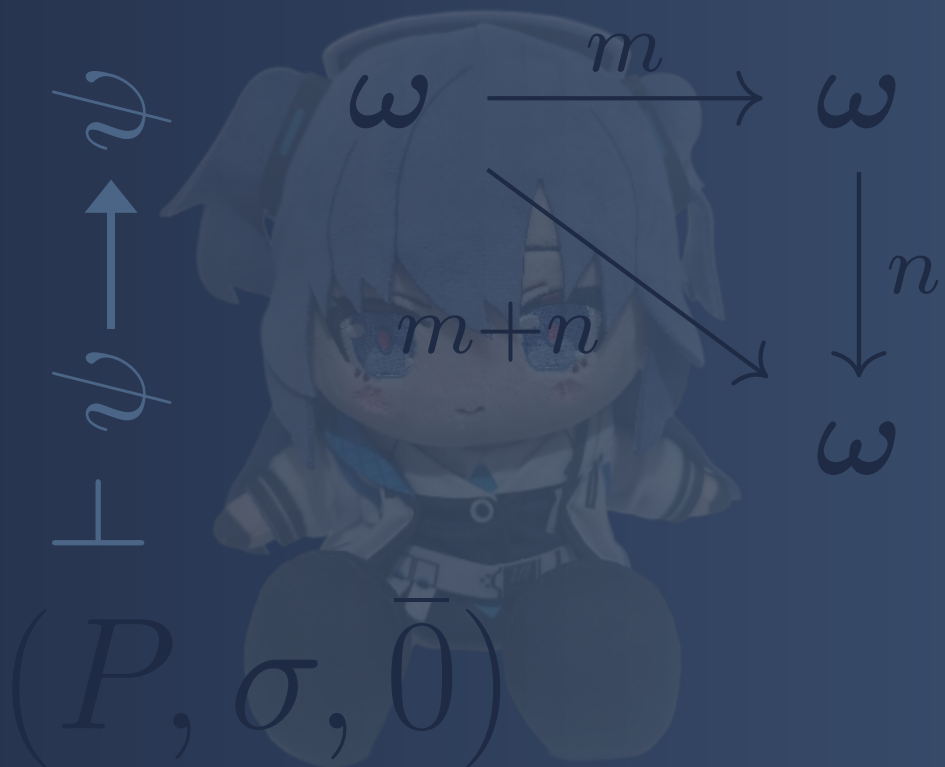


Hayase Yuuka's
notes on:

FOUNDATIONS OF MATHEMATICS



Contents

1	Logic	5
1.1	Formal languages	6
1.1.1	Language and metalanguage	6
1.1.2	First order languages	6
1.1.3	N-th order languages	6
1.2	Propositional logic	7
1.2.1	Syntactics	7
1.2.2	Semantics	7
1.2.3	General	7
1.3	First order logic	8
1.3.1	Elementary definitions	8
1.3.2	Cylindric algebras	8
1.4	Modal logic	9
1.4.1	idk	9
1.5	Other non-classical logics	10
1.6	Recursive function theory	11
1.7	Categorical logic	12
2	Set theory	13
2.1	Elements of set theory	14
2.1.1	Axiomatic systems for set theory	14
2.1.2	Classes and universes	14
2.1.3	Basic set algebra	14
2.1.4	Set theory and boolean algebras	14
2.2	Relations and functions	15
2.2.1	Relations	15
2.2.2	Functions	15
2.2.3	Pre-orders	15
2.2.4	Partial orders	15
2.2.5	Well orders	15
2.2.6	Partitions	15
2.2.7	Equivalence relations	15
2.2.8	Cartesian product	15

2.3	Natural numbers	16
2.3.1	Zermelos's naturals	16
2.3.2	Von Neumann's naturals	16
2.3.3	Uncommon naturals	16
2.3.4	Properties	16
2.3.5	The recursion theorem	16
2.3.6	Peano systems	16
2.3.7	Arithmetic	16
2.4	Extending naturals	17
2.4.1	Integers	17
2.4.2	Rings	17
2.4.3	Rationals	17
2.4.4	Ordered fields	17
2.4.5	Complete ordered fields	17
2.5	Real numbers	18
2.5.1	Cauchy sequences	18
2.5.2	Dedekind cuts	18
2.5.3	Complete ordered fields	18
2.6	Ordinals	19
2.7	Cardinality	20
2.8	Universes	21
2.9	Structures	22
3	Categories	23
3.1	Elements of categories	24
3.1.1	Axioms of category theory	24
3.1.2	Composition	24
3.1.3	Products and coproducts	24
3.1.4	Limits and colimits	24
3.1.5	Completeness	24
3.1.6	Exponentiation	24
4	Type theory	25
5	Test	27

1 Logic

1.1 Formal languages

Language and metalanguage

First order languages

N-th order languages

1.2 Propositional logic

Syntactics

Semantics

General

1.3 First order logic

Elementary definitions

Cylindric algebras

1.4 Modal logic

idk

1.5 Other non-classical logics

1.6 Recursive function theory

1.7 Categorical logic

2

Set theory

2.1 Elements of set theory

Axiomatic systems for set theory

Classes and universes

Basic set algebra

Set theory and boolean algebras

2.2 Relations and functions

Relations

Functions

Pre-orders

Partial orders

Well orders

Partitions

Equivalence relations

Cartesian product

2.3 Natural numbers

Zermelos's naturals

Von Neumann's naturals

Uncommon naturals

Properties

The recursion theorem

Peano systems

Arithmetic

2.4 Extending naturals

Integers

Rings

Rationals

Ordered fields

Complete ordered fields

2.5 Real numbers

Cauchy sequences

Dedekind cuts

Complete ordered fields

2.6 Ordinals

2.7 Cardinality

2.8 Universes

2.9 Structures

3

Categories

3.1 Elements of categories

Axioms of category theory

Composition

Products and coproducts

Limits and colimits

Completeness

Exponentiation

4 Type theory

5 Test

Exercise 8. (ii). Prove that $(f \times h) \circ (g \times k) = (f \circ g) \times (h \circ k)$

$$\begin{array}{ccccc}
 e' & \xrightarrow{k} & c & \xrightarrow{h} & d \\
 \uparrow & & \uparrow & & \uparrow \\
 e \times e' & \xrightarrow{g \times k} & a \times c & \xrightarrow{f \times h} & b \times d \\
 \downarrow & & \downarrow & & \downarrow \\
 e & \xrightarrow{g} & a & \xrightarrow{f} & b
 \end{array}$$

Proof. Consider that there's a unique arrow $(f \circ g) \times (h \circ k) : e \times e' \rightarrow b \times d$ such that the diagram:

$$\begin{array}{ccccc}
 & & e' & \xrightarrow{h \circ k} & d \\
 & \nearrow & & & \uparrow \\
 e \times e' & \xrightarrow{(f \circ g) \times (h \circ k)} & b \times d & & \\
 & \searrow & & & \downarrow \\
 & & e & \xrightarrow{f \circ g} & b
 \end{array}$$

commutes. Now, for the first diagram,

$$\begin{aligned}
 \pi_d \circ (f \times h) \circ (g \times k) &= \pi_d \circ \langle f \circ \pi_a, h \circ \pi_c \rangle \circ \langle g \circ \pi_e, k \circ \pi_{e'} \rangle \\
 &= h \circ \pi_c \circ \langle g \circ \pi_e, k \circ \pi_{e'} \rangle \\
 &= h \circ k \circ \pi_{e'}
 \end{aligned}$$

And in a similar way $\pi_b \circ (f \times h) \circ (g \times k) = f \circ g \circ \pi_e$, so the diagram:

$$\begin{array}{ccccc}
 e' & \xrightarrow{h \circ k} & d & & \\
 \uparrow & & \uparrow & & \\
 e \times e' & \xrightarrow{g \times k} & a \times c & \xrightarrow{f \times h} & b \times d \\
 \downarrow & & \downarrow & & \downarrow \\
 e & \xrightarrow{f \circ g} & b & &
 \end{array}$$

commutes. By the uniqueness of $(f \circ g) \times (h \circ k)$ we have that $(f \times h) \circ (g \times k) = (f \circ g) \times (h \circ k)$.

□