Instituto Politécnico Nacional

Escuela Superior de Física y Matemáticas

Solved Problems of Calculus IV

Integration of functions of several real variables

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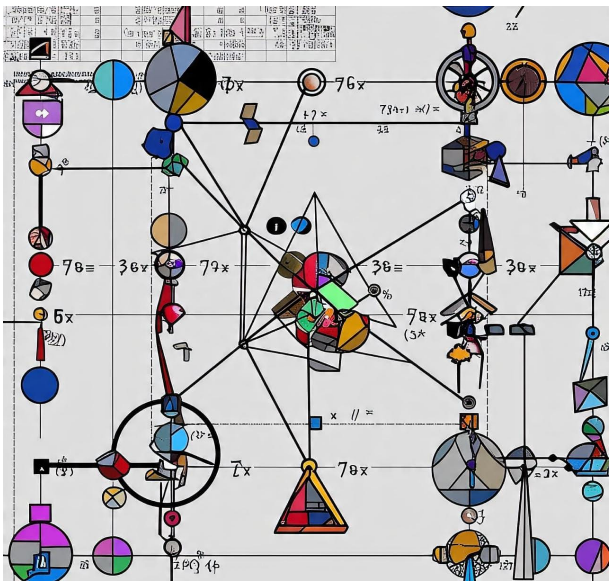

I. Disclaimer

The majority of these exercises are from [1], for practical problems (not present here) see [2]. This little text has been written in the course of the 4th semester in ESFM-IPN as entertainment for the course of Calculus IV given by Erick Lee Guzmán. Exercise *xii*) is due to Carlos Adiel Gonzales. The theory of the course can be covered using [1] and for the advanced topics like Stoke's theorem [2] is recommended to have the practical understanding and applications.

It's obvious that if the reader has to do some exercises presented here, they should attempt them first and only see the solution after an appropriate effort to resolve them.

As stated in the presentation of the page, any text presented here is not related to ESFM-IPN in any official way, the only relation is that of the authors as students of the institution at the time of the writing of these texts.

Thanks for reading :3

«А душа болит и болит и болит, где ты?

Сердце говори говорти что тебя нету»

- Регина Лисиц

II. Problems

i)

Let $f : Q \subseteq \mathbb{R}^n \to \mathbb{R}$ be bounded, $I \in \mathbb{R}$ and $\left\{P_1^k\right\}_{k\in\mathbb{N}}$, $\left\{P_2^k\right\}_{k\in\mathbb{N}}$ two successions of partitions of Q :

$$\lim_{k\to\infty} I(f,\, P_1^k) = I = \lim_{k\to\infty} S(f,\, P_2^k)$$

prove that f is integrable over Q and

$$\int_Q f = I$$

Proof.

It is clear that, $\forall\, k \in \mathbb{N}$

$$I(f,\, P_1^k) \leq \underline{\int_Q} f \leq \overline{\int_Q} f \leq S(f,\, P_2^k)$$

Therefore,

$$I = \lim_{k\to\infty} I(f,\, P_1^k) \leq \underline{\int_Q} f \leq \overline{\int_Q} f \leq \lim_{k\to\infty} S(f,\, P_2^k) = I$$

□

ii)

Prove that the following sets have measure zero:

$I)$ $A = \{a\} \subseteq \mathbb{R}^n$

$II)$ $A = \{a_1, a_2, ...\} \subseteq \mathbb{R}^n$ where $a_i \in \mathbb{R}^n$, $i \in \mathbb{N}$ y $a_i \neq a_j$ for $i \neq j$

$II)$ $A = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1],\ y = x\} \subseteq \mathbb{R}^2$

$IV)$ $\emptyset$

Proof.

$I)$ Let $\epsilon > 0$ and $a = (a_1, ..., a_n)$, considering $Q = [a_1, a_1 + \sqrt[n]{\frac{\epsilon}{2}}] \times \cdots \times [a_n, a_n + \sqrt[n]{\frac{\epsilon}{2}}]$ it is clear that $A \subseteq Q$ and $\sum v(Q) = v(Q) = \left(\sqrt[n]{\frac{\epsilon}{2}}\right)^n = \frac{\epsilon}{2} < \epsilon$.

$II)$ Let $\epsilon > 0$, from $I)$, it follows that $\forall\, a_i \in A$, $i \in \mathbb{N}$, $\exists\, Q_i : a_i \in Q$ y $v(Q_i) = \dfrac{\epsilon}{2^{i+1}}$. Hence, $A \subseteq \bigcup_{i\in\mathbb{N}} Q_i$ and $\sum v(Q_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon$.

$III)$ Let $\epsilon > 0$, by the archimedean property, $\exists\, k \in \mathbb{N} : \frac{1}{k} < \epsilon$. Thus, considering the family of sets

$$\left\{ Q_i = \left[\frac{i-1}{k}, \frac{i}{k}\right] \times \left[\frac{i-1}{k}, \frac{i}{k}\right] \right\}_{i\in[\![1,k]\!]}$$

It is given that, for $a = (x, x) \in A$, $x \in [0, 1] = \bigcup_{i=1}^{k} \left[\frac{i-1}{k}, \frac{i}{k}\right]$, consequently $\exists\, l \in [\![1, k]\!] : x \in \left[\frac{l-1}{k}, \frac{l}{k}\right]$, and $a \in \left[\frac{l-1}{k}, \frac{l}{k}\right] \times \left[\frac{l-1}{k}, \frac{l}{k}\right]$, thus $A \subseteq \bigcup_{i=1}^{k} Q_i$.

Moreover, $v(Q_i) = \frac{1}{k^2}, \forall\, i \in [\![1, k]\!]$, therefore $\sum_{i=1}^{k} v(Q_i) = \sum_{i=1}^{k} \frac{1}{k^2} = \frac{1}{k} < \epsilon$.
Note: sketch this family graphically.

IV) Let $\epsilon > 0$, any rectangle or family of rectangles with $\sum v(Q) < \epsilon$ suffices, in particular Q given in *I*).

□

iii)

Prove the following properties of measure zero sets.
1. Let $\{A_i\}_{i\in\mathbb{N}}$ a numerable collection of measure zero sets, then $\bigcup_{i\in\mathbb{N}} A_i$ has measure zero.
2. $A \subseteq \mathbb{R}^n$ has measure zero $\iff$ $\forall\, \epsilon > 0$, $\exists\, \{Q_i\}_{i\in\mathbb{N}}$:

   I) $A \subseteq \bigcup_{i\in\mathbb{N}} \mathring{Q}_i$

   II) $\sum_{i\in\mathbb{N}} v(Q_i) < \epsilon$

Proof.

1. Let $\epsilon > 0$, $\forall\, i \in \mathbb{N}$ let $\{Q_{ij}\}_{j\in\mathbb{N}} : A_i \subseteq \bigcup_{j\in\mathbb{N}} Q_{ij}$ and $\sum_{j\in\mathbb{N}} v(Q_{ij}) < \frac{\epsilon}{2^{i+1}}$. Then

$$\bigcup_{i\in\mathbb{N}} A_i \subseteq \bigcup_{i,j\,\in\mathbb{N}} Q_{ij}$$

and

$$\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} v(Q_{ij}) < \sum_{i\in\mathbb{N}} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon$$

2. $\Longleftarrow$ It is clear that the same collection $\{Q_i\}_{i\in\mathbb{N}}$ works.
   $\Longrightarrow$ Let $\epsilon > 0$, since A has measure zero, $\exists\, \{K_i\}_{i\in\mathbb{N}} : A_i \subseteq \bigcup_{i\in\mathbb{N}} K_i$ and $\sum_{i\in\mathbb{N}} v(K_i) < \frac{\epsilon}{2}$. Now, each K_i has the form

$$K_i = [a_{i1}, b_{i1}] \times \cdots \times [a_{in}, b_{in}] \subseteq \mathbb{R}^n$$

By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q_{im}$, $m \in [\![1, n]\!] : q_{im} \in (0, b_{im} - a_{im}]$.

Therefore, considering the collection $\{Q_i\}_{i\in\mathbb{N}}$ with

$$Q_i = \left[a_{i1} - \frac{q_{i1}}{2}, b_{i1} + \frac{q_{i1}}{2}\right] \times \cdots \times \left[a_{in} - \frac{q_{in}}{2}, b_{in} + \frac{q_{in}}{2}\right] \subseteq \mathbb{R}^n$$

it follows that

$$A \subseteq \bigcup_{i\in\mathbb{N}} K_i \subseteq \bigcup_{i\in\mathbb{N}} \left(a_{i1} - \frac{q_{i1}}{2}, b_{i1} + \frac{q_{i1}}{2}\right) \times \cdots \times \left(a_{in} - \frac{q_{in}}{2}, b_{in} + \frac{q_{in}}{2}\right) = \bigcup_{i\in\mathbb{N}} \mathring{Q}_i$$

Moreover,

$$\begin{aligned}\sum_{i\in\mathbb{N}} v(Q_i) &= \sum_{i\in\mathbb{N}} \prod_{m=1}^{n} (b_{im} - a_{im} + q_{im}) \\ &\leq \sum_{i\in\mathbb{N}} \prod_{m=1}^{n} 2(b_{im} - a_{im}) = 2\sum_{i\in\mathbb{N}} v(K_i) < \epsilon\end{aligned}$$

□

Note: This exercise should appear intuitive to the reader, even more, the explicit construction in the proof may seem (and is) redundant. However, it may be instructive to the reader to see it if he has not done it before.

iv)

Let $\varphi : [0,1] \to \mathbb{R}$ be continuous, prove that the graph of the function $\Gamma = \left\{(x,y) \in \mathbb{R}^2 : x \in [0,1], y = \varphi(x)\right\}$ has measure zero

Proof.

Let $\epsilon > 0$, since $[0,1]$ is compact, φ is uniformly continuous, then $\exists\, \delta > 0 : \forall\, x, y \in [0,1]$, $|x-y| < \delta \implies |\varphi(x) - \varphi(y)| < \epsilon$.

Now, by the archimedean property, $\exists\, k \in \mathbb{N} : \frac{1}{k} < \delta$. Considering

$$Q_i := \underbrace{\left[\frac{i-1}{k}, \frac{i}{k}\right]}_{I_i} \times \left[m(f, I_i), M_(f, I_i)\right], \qquad i \in [\![1,k]\!]$$

given $x \in [0,1] = \bigcup_{i\in[\![1,k]\!]} I_i$, it is clear that $(x, \varphi(x)) \in \bigcup_{i\in[\![1,k]\!]} \left(I_i \times \left[m(f,I_i), M_(f,I_i)\right]\right)$, thus $\Gamma \subseteq \bigcup_{i\in[\![1,k]\!]} Q_i$.

On the other hand, by the continuity of φ, $\forall\, M(f,I_i), m(f,I_i)$, $i \in [\![1,k]\!]$, $\exists\, x_{0i}, y_{0i} \in I_i : f(x_{0i}) = M(f,I_i), f(y_{0i}) = m(f,I_i)$ and $|x_{0i} - y_{0i}| < \dfrac{1}{k} < \delta \implies |M(f,I_i) - m(f,I_i)| < \epsilon$.

Therefore,

$$\sum_{i\in[\![1,k]\!]} v(Q_i) = \sum_{i\in[\![1,k]\!]} \frac{1}{k}\left[M(f,I_i) - m(f,I_i)\right] < \epsilon$$

□

v)

Prove that the following function is not integrable over Q

$$f : \underbrace{[0,1]\times[0,1]}_{Q} \to \mathbb{R}$$

$$(x,y) \mapsto \begin{cases} y, & x \in \mathbb{I} \\ 1/2, & x \in \mathbb{Q} \end{cases}$$

Proof.

First, let us show that the set of discontinuities of f is $D = Q \setminus \{(x, y) \in \mathbb{R}^2 : y = 1/2, x \in [0, 1]\} = Q \setminus C$.

f is not continuous on D, for if $x_0 \in \mathbb{Q}$

$$\lim_{n\to\infty}\left(x_0 + \frac{1}{n}, y_0\right) = (x_0, y_0) = \lim_{n\to\infty}\left(x_0 + \frac{\sqrt{2}}{n}, y_0\right)$$

$$\lim_{n\to\infty} f\left(x_0 + \frac{1}{n}, y_0\right) = \frac{1}{2} \neq y_0 = \lim_{n\to\infty} f\left(x_0 + \frac{\sqrt{2}}{n}, y_0\right)$$

and if $x_o \in \mathbb{I}$, by the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists \{q_n\}_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty}\left(x_0 + \frac{1}{n}, y_0\right) = (x_0, y_0) = \lim_{n\to\infty}(q_n, y_0)$$

$$\lim_{n\to\infty} f\left(x_0 + \frac{1}{n}, y_0\right) = y_0 \neq \frac{1}{2} = \lim_{n\to\infty} f(q_n, y_0)$$

And f is continuous on C, for given $\epsilon > 0$, $a = (x_0, 1/2) \in C$, $\exists\, \delta = \epsilon$: if $b = (x, y) \in Q$ then $|f(b) - f(a)| = |y - \frac{1}{2}| < ||b - a|| < \delta = \epsilon$. (if $x \in \mathbb{Q}$ $0 < \epsilon$)

Now, D does not have measure zero, otherwise $Q = D \cup C$ would have measure zero, for C is the graph of the constant function $f(x) = 1/2$ on $[0, 1]$. Therefore, by Lebesgue's criterion, f is not integrable

□

vi)

Let $f: \ Q \to \mathbb{R}$, where $Q \subseteq \mathbb{R}^2$ is a rectangle, f bounded and integrable over Q. Determine if $G_f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y) \ \wedge \ (x, y) \in Q\}$ has masure zero.

Proof.

Since f is integrable over Q, given $\epsilon > 0$, $\exists\, P$ partition of Q : $S(f, P) - I(f, P) < \epsilon$, where $P = (P_1, P_2)$ and $P_1 = \{t_i\}_{i\in[\![0,k_1]\!]}, P_2 = \{l_j\}_{j\in[\![0,k_2]\!]}$, and the subrectangle R_{ij} determined by P is given by $R_{ij} = [t_i, t_{i+1}] \times [l_i, l_{i+1}]$, $(i, j) \in [\![0, k_1]\!] \times [\![0, k_2]\!]$.

Thus, considering $Q_{ij} = R_{ij} \times [m(f, R_{ij}), M(f, R_{ij})]$, we have that if $(x, y) \in Q = \bigcup_{R_{ij}\in P} R_{ij}$, then, $G_f \subseteq \bigcup Q_{ij}$. Moreover,

$$\begin{aligned}\sum v(Q_{ij}) &= \sum_{R_{ij}\in P} v(R_{ij})[M(f, R_{ij}) - m(f, R_{ij})] \\ &= \sum_{R_{ij}\in P} [M(f, R_{ij})]v(R_{ij}) + \sum_{R_{ij}\in P} [m(f, R_{ij})]v(R_{ij}) \\ &= S(f, P) - I(f, P) < \epsilon\end{aligned}$$

$\therefore \quad G_f$ has measure zero

□

vii)

Let $A \subseteq \mathbb{R}^2$ be open, $f : A \to \mathbb{R}$ be a function of class C^2 and $Q \subseteq A$ a rectangle.

a) Using Fubini's theorem and the fundamental theorem of calculus prove that

$$\int_Q \frac{\partial^2 f}{\partial x_2 \partial x_1} = \int_Q \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

b) Using *a*), prove that $\frac{\partial^2 f}{\partial x_2 \partial x_1}(a) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(a), \forall\, a \in A.$

Proof.

a) Let $Q = [a, b] \times [c, d]$, on one hand,

$$\begin{aligned}\int_Q \frac{\partial^2 f}{\partial x_2 \partial x_1} &= \int_a^b \int_c^d \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_1, x_2) dx_2 dx_1 = \int_a^b \frac{\partial f}{\partial x_1}(x_1, d) - \frac{\partial f}{\partial x_1}(x_1, c) \\ &= [f(b, d) - f(a, d)] - [f(b, c) - f(a, c)] \\ &= f(a, c) - f(a, d) + f(b, d) - f(b, c)\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_Q \frac{\partial^2 f}{\partial x_1 \partial x_2} &= \int_c^d \int_a^b \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) dx_1 dx_2 = \int_c^d \frac{\partial f}{\partial x_2}(b, x_2) - \frac{\partial f}{\partial x_2}(a, x_2) \\ &= [f(b, d) - f(b, c)] - [f(a, d) - f(a, c)] \\ &= f(a, c) - f(a, d) + f(b, d) - f(b, c)\end{aligned}$$

b) Suppose that $\exists\, a \in A : \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) \neq \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)$, without loss of generality, assume $\frac{\partial^2 f}{\partial x_2 \partial x_1}(a) > \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)$. Since f is of class C^2 and A is open, $\exists\, U \subseteq A$ open neighborhood of $a : \forall\, x \in U$, $\frac{\partial^2 f}{\partial x_2 \partial x_1}(x) > \frac{\partial^2 f}{\partial x_1 \partial x_2}(x)$. Now, we can choose a rectangle $Q \subseteq U$ with $a \in Q$, and again since f is of class C^2, integrating over $Q \subseteq A$, $\int_Q \frac{\partial^2 f}{\partial x_2 \partial x_1} > \int_Q \frac{\partial^2 f}{\partial x_1 \partial x_2}$, contradicting *a*).

□

viii)

Let $Q = [-1, 1] \times [0, 1] \subseteq \mathbb{R}^2$, prove the existence and find the value of $\int_Q |y - x|$.

Proof.

It is clear that $\begin{aligned} f : \quad & Q \to \mathbb{R} \\ & (x, y) \mapsto |y - x| \end{aligned}$ is continuous on Q so it is integrable over Q. Using Fubini's theorem,

$$\int_Q |y - x| = \int_{-1}^0 \int_0^1 (y - x) dy dx \; + \int_0^1 \left(\int_0^x (x - y) dy \; + \int_x^1 (y - x) dy \right) dx \underbrace{=}_{\text{boring}} \frac{4}{3}$$

□

ix)

Let $A \subseteq \mathbb{R}^k$, $B \subseteq \mathbb{R}^n$ be rectangles and $f : A \times B \to \mathbb{R}$ bounded. Determine if $\exists \int_Q f \Rightarrow \exists D \subseteq \mathbb{R}^k$ of measure zero : $\exists \int_{y\in B} f(x,y)dy, \ \forall x \in A \setminus D$.

Proof.

Since $\exists \int_Q f$, $\int_{x\in A} \underline{\int_{y\in B}} f(x,y)dydx = \int_{x\in A} \underline{I}(x)dx = \int_{x\in A} \overline{I}(x)dx = \int_{x\in A} \overline{\int_{y\in B}} f(x,y)dydx$. Now, it is clear that $\overline{I}(x) - \underline{I}(x) \geq 0 \ \forall x \in A$, and $\int_{x\in A} [\overline{I}(x) - \underline{I}(x)]dx = 0$. Thus, $\exists D \subseteq \mathbb{R}^k$ of measure zero : $\overline{I}(x) - \underline{I}(x) = 0 \ \ \forall x \in A \setminus D$.

$$\therefore \quad \exists \int_{y\in B} f(x,y)dy, \ \ \forall x \in A \setminus D$$

□

x)

Let $\{Q_i\}_{i\in}$ be a countable number of rectangles such that $Q \subseteq \bigcup_{i\in\mathbb{N}} Q_i$. Prove that $v(Q) \leq \sum_{i\in\mathbb{N}} v(Q_i)$.

Proof.

Let $Q_i = [a_1^i, b_1^i] \times \cdots \times [a_n^i, b_n^i]$, define $R_i(\delta) = \left[a_1^i - \frac{\delta}{2}, b_1^i + \frac{\delta}{2}\right] \times \cdots \times \left[a_n^i - \frac{\delta}{2}, b_n^i + \frac{\delta}{2}\right]$. Now, it is clear that $Q \subseteq \bigcup_{i\in\mathbb{N}} Q_i \subseteq \bigcup_{i\in\mathbb{N}} \mathring{R}_i(\delta)$, since Q is compact, $\exists \left\{\mathring{R_i}(\delta)\right\}_{i\in[\![1,N]\!]}$ finite cover of Q. Setting $A_j = b_j^i - a_j^i$, we have that $v(R_i(\delta)) = \prod_{j=1}^{n} (A_j^i + \delta) = \prod_{j=1}^{n} A_j + P(\delta) = v(Q_i) + P(\delta)$.

Where $P(\delta) = \sum_{j=1}^{n} c_j \delta^j$ is a polynomial of degree n, so it is continuous, particularly in 0, then $\forall \epsilon_1 > 0, \exists \gamma > 0$: if $0 < \delta < \gamma \ \Rightarrow \ |P(\delta) - P(0)| = P(\delta) < \epsilon_1$. In particular, for $\epsilon_1 = \frac{\epsilon}{2^i} > 0, \exists \gamma_i > 0$: if $0 < \delta < \gamma_i \ \Rightarrow \ P(\delta) < \frac{\epsilon}{2^i}$.

Finally, given that $Q \subseteq \bigcup_{i\in\mathbb{N}} R_i(\delta)$, then

$$v(Q) \leq \sum_{i=1}^{N} v(R_i(\delta)) \leq \sum_{i\in\mathbb{N}} v(R_i(\delta)) = \sum_{i\in\mathbb{N}} \left[v_(Q_i) + P(\delta)\right] < \sum_{i\in\mathbb{N}} \left[v_(Q_i) + \frac{\epsilon}{2^i}\right] = \sum_{i\in\mathbb{N}} v_(Q_i) + \epsilon$$

$$\therefore \quad v(Q) \leq \sum_{i\in\mathbb{N}} v(Q_i)$$

□

xi)

Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be increasing. Prove that f has at most a countable set of discontinuities

Proof.

Since f is increasing, $\forall x \in (a,b)$, $\exists\, f(x^-) = \sup\{f(t) : t < x\}, f(x^+) = \inf\{f(t) : t > x\}$ and f is continuous at $x \iff f(x^-) = f(x) = f(x^+)$. Thus, let $D = \{x \in (a,b) : f(x^-) < f(x^+)\}$ be the set of discontinuities and $I_x = (f(x^-), f(x^+))$, if $x, y \in D$ with $x < y$, since f is increasing, $f(x^+) \leq (y^-)$ so $\{I_x\}_{x \in D}$ is a disjoint family. Therefore, by the density of $\mathbb{Q}$ in $\mathbb{R}$, $\forall x \in D$, $\exists\, q_x \in I_x \cap \mathbb{Q}$: $\begin{array}{rl} \varphi : & D \to \mathbb{Q} \\ & x \mapsto q_x \end{array}$ is injective, from which the result follows.

□

xii)

Let $A = \{x \in [0,1] : x = p/q \wedge p, q \in \mathbb{Q}^+ \cup \{0\} \wedge q \neq 0 \wedge (p,q) = 1\}$ and $Q = [0,1] \times [0,1]$. Let

$$\begin{array}{rl} f: & Q \to \mathbb{R} \\ & (x,y) \mapsto \begin{cases} 1/q, & (x,y) \in A \times \mathbb{Q} \\ 0 \quad , & (x,y) \in (\mathbb{I} \times [0,1]) \cup (A \times \mathbb{I}) \end{cases} \end{array}$$

I) Prove that $\exists \displaystyle\int_Q f$

II) Calculate $\displaystyle\overline{\int_{y\in[0,1]}} f(x,y)dy \quad \wedge \quad \underline{\int_{y\in[0,1]}} f(x,y)dy$

III) Verify Fubini's Theorem

Proof.

I) Let $\epsilon > 0$, $\exists\, k \in \mathbb{N} : \frac{1}{k} < \epsilon$, moreover, there are a finite number of rational numbers : $x = p/q \in A$, $q < k$, i.e., $\frac{1}{k} < \frac{1}{q}$.

Now, let $(x_0, y_0) \in Q \cap [(\mathbb{I} \times [0,1]) \cup (A \times \mathbb{I})]$ and $\delta := \dfrac{\min\{|x - x_0| : x \in A,\ x = p/q \wedge \frac{1}{k} < \frac{1}{q}\}}{2}$, if $(x,y) \in Q : ||(x,y) - (x_0,y_0)|| < \delta$, then

a) if $(x,y) \in Q \cap [(\mathbb{I} \times [0,1]) \cup (A \times \mathbb{I})]$, then $|f(x,y) - f(x_0,y_0)| = 0 < \epsilon$

b) if $(x,y) \in (A \times \mathbb{Q}) \cap Q$ with $x = p/q$, then $|f(x,y) - f(x_0,y_0)| = \frac{1}{q} \leq \frac{1}{k} < \epsilon$

Therefore, f is continuous on $Q \cap [(\mathbb{I} \times [0,1]) \cup (A \times \mathbb{I})]$, so f is discontinuous at most on $D = (\mathbb{Q}^+ \cup \{0\}) \times (\mathbb{Q}^+ \cup \{0\}) \cap Q \subseteq \mathbb{Q} \times \mathbb{Q} \hookrightarrow \mathbb{R}^2$ which is countable, for the product of countable sets is countable, moreover $D \hookrightarrow \mathbb{R}^2$ so it has measure zero and by Lebesgue's theorem $\exists \displaystyle\int_Q f$.

II) If $x \in \mathbb{I} \Rightarrow \forall P$ partition of Q, $R \in P$, $m(f,R) = 0 = M(f,R) \Rightarrow \displaystyle\overline{\int_{y\in[0,1]}} f(x,y)dy = 0 = \underline{\int_{y\in[0,1]}} f(x,y)dy$.

If $x \in A \Rightarrow \forall P$ partition of Q, $R \in P$, $\exists\, y_1 \in \mathbb{Q}^+ \cup \{0\} \wedge y_2 \in \mathbb{I}$, $\Rightarrow m(f,R) = 0, M(f,R) = 1/q$. Then $\displaystyle\overline{\int_{y\in[0,1]}} f(x,y)dy = \begin{cases} 1/q, x \in A \\ 0 \quad , x \in \mathbb{I} \end{cases}$, $\displaystyle\underline{\int_{y\in[0,1]}} f(x,y)dy = 0$.

III) $\forall P$ partition of Q, $\exists\, (l,k) \in \mathbb{I} \times [0,1] \Rightarrow m(f,R) = 0 \Rightarrow I(f,P) = 0$, so $\displaystyle\int_Q f = 0$. Now,

$$\int_{x\in[0,1]} \underline{\int_{y\in[0,1]}} f(x,y) = \int_{x\in[0,1]} 0 = 0 \quad \wedge \quad \int_{x\in[0,1]} \overline{\int_{y\in[0,1]}} f(x,y)dy = \int_{x\in[0,1]} h(x) = 0 \quad \text{(Thomae's function)}$$

$$\therefore \quad \int_Q f = \int_{x\in[0,1]} \underline{\int_{y\in[0,1]}} f(x,y) = \int_{x\in[0,1]} \overline{\int_{y\in[0,1]}} f(x,y)dy$$

□

xiii)

Let $A \subseteq \mathbb{R}^n$ open and bounded, $f : A \to \mathbb{R}^n$ continuous, bounded, nonnegative, and $A \subseteq Q$ a rectangle. Prove that

$$I\int_A f = \underline{\int_Q} f_A$$

Proof.

Let P be a partition of Q and R be a sub rectangle of Q determined by P. Since each R is compact and rectifiable, so is $C = \bigcup_{R\subseteq A} R$. Now, since P is a partition, $R_a \cap R_b = \emptyset \iff R_a \neq R_b, \quad R_a, R_b \in P$. Finally, remembering the following

$$\int_\emptyset f = \int_Q f_\emptyset = \int_Q 0 = 0, \quad \int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f$$

, first note $f(x) \geq m(f,R), \forall x \in R$ so

$$\int_R f \geq \int_R m(f,R) = m(f,R)\int_R = m(f,R)v(R)$$
$$\implies \int_C f = \sum_{R\subseteq A}\int_R f \geq \sum_{R\subseteq A} m(f,R)v(R) = \sum_{R\subseteq A} m(f_A,R)v(R) = I(f_A,P)$$
$$\therefore \quad I\int_A f \geq \int_C f \geq \underline{\int_Q} f_A$$

On the other hand, let $C \subseteq A$ be compact and rectifiable, since f is continuous over A, in particular it's continuous on C, so $\exists \int_C f = \int_Q f_C$. It's clear that $f_A(x) \geq f_C(x), \quad \forall x \in Q$. Thus, given a partition P of Q and R a sub rectangle of Q determined by P, $m(f_A,R) \geq m(f_C,R)$,

$$\implies I(f_A,P) = \sum_{R\in P} m(f_A,R)v(R) \geq \sum_{R\in P} m(f_C,R)v(R) = I(f_C,P)$$
$$\implies \underline{\int_Q} f_A \geq I(f_A,P) \geq \underline{\int_Q} f_C = \int_Q f_C = \int_C f$$
$$\implies \underline{\int_Q} f_A \geq I\int_A f$$

$\therefore$

$$I\int_A f = \underline{\int_Q} f_A$$

□

xiv)

Let $A \subseteq \mathbb{R}^{n-1}$ open and rectifiable, $p \in \mathbb{R}^n$ with $p_n > 0$. Consider $S = \{x \in \mathbb{R}^n : x = (1-t)a + tp, \quad a \in A \times \{0\}, t \in (0,1)\}$, find $v(S)$ in terms of $v(A)$.

Define the function $\begin{array}{rl}\alpha: & \underbrace{A\times(0,1)}_{U}\subseteq\mathbb{R}^n\to S\\ & (a,t)\mapsto(1-t)a+tp=((1-t)a_1+tp_1,...,(1-t)a_{n-1}+tp_{n-1},tp_n)\end{array}$ It's clear that U is open, α is bijective over its image and it's of class $C^1(U)$. Let's see that α is non-singular, i.e, $\det D\alpha(u)\neq 0,\ \forall u\in U$.

$$[D\alpha(a,t)]_\beta=\left(\frac{\partial\alpha_m}{\partial x_n}(a,t)\right)_{m,n\in[\![1,n]\!]}$$
$$=\begin{pmatrix}1-t & 0 & \cdots & 0 & p_1-a_1\\ 0 & 1-t & \cdots & 0 & p_2-a_2\\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1-t & p_{n-1}-a_{n-1}\\ 0 & 0 & \cdots & 0 & p_n\end{pmatrix}$$

Hence, $\det D\alpha(a,t)=(1-t)^{n-1}p_n>0,\ \forall u\in U$.

Using the theorem of change of variable and Fubini's theorem for bounded regions,

$$v(S)=\int_S 1=\int_{A\times(0,1)}1\circ\alpha|\det D\alpha|=\int_A 1\int_{(0,1)}(1-t)^{n-1}p_n=v(A)\frac{p_n}{n}$$

□

xv)

Let $\begin{array}{rl}f: & \mathbb{R}\to\mathbb{R}\\ & x\mapsto x\end{array}$ show that given $\lambda\in\mathbb{R}$, $\exists\{C_N\}_{N\in\mathbb{N}}$ family of compact rectificable subsests of $\mathbb{R}: C_N\subseteq \mathring{C}_{N+1},\ \bigcup_{N\in\mathbb{N}}C_N=\mathbb{R}$, and $\lim_{N\to\infty}\int_{C_N}f=\lambda$. $\exists\ I\int_{\mathbb{R}}f$?

Proof.

For $\lambda\geq 0$, consider $C_N=[-\sqrt{N},\sqrt{N+2\lambda}]$. It's clear that $C_N\subseteq\mathring{C}_{N+1},\ \bigcup_{N\in\mathbb{N}}C_N=\mathbb{R}$. And, $\int_{C_N}f=\int_{-\sqrt{N}}^{\sqrt{N+2\lambda}}x\,dx=\frac{N+2\lambda}{2}-\frac{N}{2}=\lambda$. It's similar for $\lambda<0$ by taking $C_N=[-\sqrt{N-2\lambda},\sqrt{N}]$. In any case $\lim_{N\to\infty}\int_{C_N}f=\lambda$. Now, $\left\{\int_C|f|: C\subseteq\mathbb{R}\text{ is compact and rectificable}\right\}$ is not bounded, for it suffices to see that $\lim_{N\to\infty}\int_0^{\sqrt{2N}}x\,dx=\lim_{N\to\infty}N=+\infty$. $\therefore\nexists I\int_{\mathbb{R}}f$

□

xvi)

Let $A\subseteq\mathbb{R}$ be open and $f,g:A\to\mathbb{R}$ continuous, suppose that $|f(x)|\leq g(x),\ \forall x\in A$. Prove that if $\exists\ I\int_A g\implies\exists\ I\int_A f$.

Proof.

Let $C \subseteq A$ be compact and rectificable, since f and g are continuous, by monotony of the integral

$$\int_C |f| \leq \int_C g \leq \int_C |g|$$

since $\exists\ I \displaystyle\int_A g$, $\exists \sup \mathscr{C}_g = \sup \left\{ \displaystyle\int_C |g| : C \subseteq A \text{ is compact and rectificable} \right\}$ and $\displaystyle\int_C |f| \leq \sup \mathscr{C}_g$. Therefore $\mathscr{C}_f$ is bounded and $\exists\ I \displaystyle\int_A f$.

□

xvii)

Let $A \subseteq \mathbb{R}^n$ be open. $f : A \to \mathbb{R}^n$ is said to be locally bounded on A if $\forall\ x \in A$, $\exists\ V_\delta(x) : |f(x)| \leq M$, $\forall\ x \in A$, for some $M \in \mathbb{R}$. Let $\mathscr{F}(A) = \{ f : A \to \mathbb{R}^n : f$ is locally bounded on A and continuoues except on a set D of meazurezero$\}$.

$I)$ Prove that if f is continuous on $A \implies f \in \mathscr{F}(A)$

$II)$ Show that if $f \in \mathscr{F}(A) \implies f$ is bounded $\forall\, C \subseteq A$ compact.

Proof.

$I)$ Since A is open, $\forall\, x \in A$, $\exists\, \delta > 0 : V_\delta(x) \subseteq A$. Now consider $C = \overline{V_{\delta/2}(x)} \subseteq A$ compact. By the continuity of f, $\exists\, M \in C : |f(x)| \leq M, \forall\, x \in C$, i.e., $f \in \mathscr{F}(A)$.

$II)$ Let $C \subseteq A$ be compact, since $f \in \mathscr{F}(A)$, $\forall\, x \in C, \exists\, V_{\delta_x}(x) : C \subseteq \displaystyle\bigcup_{x \in C} V_{\delta_x}(x)$, since C is compact, $C \subseteq \displaystyle\bigcup_{i=1}^{N} V_{\delta_{x_i}}(x)$. Thus, letting $M = \max\{M_{\delta_{x_i}} : i \in [\![1, n]\!]\} \implies |f(x)| \leq M,\ \forall\, x \in C$.

□

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